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Accelerating random walks by disorder

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Abstract. We investigate the dynamic impact of heterogeneous environments on superdiffusive random walks known as Lévy flights. We devote particular attention to the relative weight of source and target locations on the rates for spatial displacements of the random walk. Unlike ordinary random walks which are slowed down for all values of the relative weight of source and target, non-local superdiffusive processes show distinct regimes of attenuation and acceleration for increased source and target weight, respectively. Consequently, spatial inhomogeneities can facilitate the spread of superdiffusive processes, in contrast to the common belief that external disorder generally slows down stochastic processes. Our results are based on a novel type of fractional Fokker–Planck equation which we investigate numerically and by perturbation theory for weak disorder.

A particle performing ordinary diffusion is typically characterized by a spatiotemporal scaling relation $|\boldsymbol{X}(t)| \sim t^{1/2}$. An increasing number of physical and biological systems are in conflict with this relation and exhibit anomalous diffusion. Whenever the spatiotemporal scaling relation $|\boldsymbol{X}(t)| \sim t^{1/\mu}$ with an exponent $0 < \mu < 2$ a system is said to exhibit superdiffusive behaviour. Superdiffusion has been discovered in a wide range of systems, for instance chaotic dynamical systems [1], particles in turbulent flows [2], saccadic eye movements [3], trajectories of foraging animals [4], and very recently in the geographic dispersal of bank notes [5].

Theoretically, superdiffusion is often accounted for by scale free random walks known as Lévy flights for which successive spatial displacements $\Delta\boldsymbol{x}$ are drawn from a probability density function (pdf) with an algebraic tail, i.e., $p(\Delta\boldsymbol{x}) \sim |\Delta\boldsymbol{x}|^{-d-\mu}$, where d is the spatial dimension

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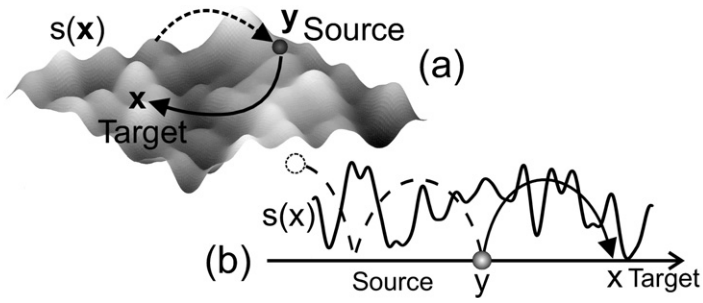

Figure 1. Random walk processes in inhomogeneous salience fields $s(\boldsymbol{x})$ in two (a) and one (b) dimensions. Source and target locations of a random jump are denoted by $\boldsymbol{y}$ and $\boldsymbol{x}$, respectively.

and the Lévy exponent $\mu < 2$, such that the variance of $\Delta\boldsymbol{x}$ is divergent [6]. Embedded in the context of continuous time random walks (CTRW) one can show that the dynamics of the pdf of finding a superdiffusive particle at position $\boldsymbol{x}$ at time t is governed by a fractional generalization of the diffusion equation

$$\partial_t p(\boldsymbol{x}, t) = D_\mu \Delta^{\mu/2} p(\boldsymbol{x}, t), \tag{1}$$

in which the ordinary Laplacian Δ is replaced by the fractional operator $\Delta^{\mu/2}$, a non-local integral operator, the action of which is equivalent to a multiplication by $-|\boldsymbol{k}|^\mu$ in Fourier space. The constant D_μ is the generalized diffusion coefficient.

A process governed by (1) is spatially homogeneous and isotropic, as the probability rate $w(\boldsymbol{y}|\boldsymbol{x})$ of a displacement from $\boldsymbol{x}$ to $\boldsymbol{y}$ depends only on distance, i.e., $w(\boldsymbol{y}|\boldsymbol{x}) \propto |\boldsymbol{x} - \boldsymbol{y}|^{-(\mathrm{d}+\mu)}$.

Numerous random processes, however, occur on spatially disordered substrates or evolve in the presence of quenched spatial inhomogeneities. Depending on the underlying physical or biological system, one obtains various generalizations of (1) which incorporate the spatial structure. In the generalized Langevin approach [7]–[9] an additional force term $-\nabla F$ on the right hand side of (1) accounts for an external position dependent force field $F(\boldsymbol{x})$. In topologically superdiffusive systems, such as intersegment transfer of gene regulatory enzymes on DNA strands [10]–[12], the transition rate is modified by a Boltzmann factor, i.e., $w(\boldsymbol{y}|\boldsymbol{x}) \propto |\boldsymbol{x} - \boldsymbol{y}|^{-(\mathrm{d}+\mu)} \times \exp -\beta[V(\boldsymbol{y}) - V(\boldsymbol{x})]/2$, where $V(\boldsymbol{x})$ is a position dependent potential and β is the inverse temperature. In subordinated superdiffusive processes, an ordinary diffusion process subjected to an external force is sampled at highly variable operational time intervals [13]. All three systems exhibit very different response properties to the imposed spatial structure [14], but converge to the same Fokker–Planck equation in the limit of ordinary diffusion.

Here, we investigate the dynamic impact of heterogeneous environments and devote particular attention to the relative impact of source and target locations on the rates $w(\boldsymbol{x}|\boldsymbol{y})$ for spatial displacements of the random walk (see figure 1). We define the spatial inhomogeneity in terms of the attractivity or salience $s(\boldsymbol{x}) > 0$ of a location $\boldsymbol{x}$. For large and small values of $s(\boldsymbol{x})$, the location $\boldsymbol{x}$ is attractive and unattractive, respectively. We assume that in equilibrium a walker's stationary probability $p^\star(\boldsymbol{x})$ of being at a location $\boldsymbol{x}$ is proportional to the salience at $\boldsymbol{x}$, i.e., $p^\star(\boldsymbol{x}) \propto s(\boldsymbol{x})$. In that respect, the salience field can be defined operationally as the likelyhood of finding a walker at site $\boldsymbol{x}$. Furthermore, we assume that a transition from $\boldsymbol{y}$ to $\boldsymbol{x}$ is more likely to occur when the salience is large at the target location $\boldsymbol{x}$, and less likely to occur when the salience is large at the source location $\boldsymbol{y}$. This leads to a transition rate

$$w(\boldsymbol{x}|\boldsymbol{y}) = \frac{1}{\tau} s^c(\boldsymbol{x}) f(|\boldsymbol{x}-\boldsymbol{y}|) s^{c-1}(\boldsymbol{y}), \tag{2}$$

where τ is a time constant and the sandwiched term $f(|\boldsymbol{x}-\boldsymbol{y}|) \propto |x-y|^{-(\mathrm{d}+\mu)}$ with $0 < \mu \leqslant 2$ accounts for Lévy flight jump lengths. Inserting (2) into the master equation

$$\partial_t p(\boldsymbol{x},t) = \int \mathrm{d}\boldsymbol{y} w(\boldsymbol{x}|\boldsymbol{y}) p(\boldsymbol{y},t) - w(\boldsymbol{y}|\boldsymbol{x}) p(\boldsymbol{y},t), \tag{3}$$

one can see that detailed balance is fullfilled and that the stationary solution if it exists is proportional to the salience. The central parameter in our analysis is the weight parameter $0 \leqslant c \leqslant 1$, which quantifies the relative impact of source and target salience on the dynamics. When $c = 1$ a transition $\boldsymbol{y} \to \boldsymbol{x}$ only depends on the salience at the target site and is independent of the salience at the source, when $c = 0$ the salience of the target site has no influence on the transition and the rate is decreased with increasing salience at the source. The intermediate case $c = 1/2$ is equivalent to topological superdiffusion approach with a salience given by a Boltzmann factor $s(\boldsymbol{x}) = \mathrm{e}^{-\beta V(\boldsymbol{x})}$.

In combination with the rate (2) the master equation (3) is equivalent to the fractional Fokker–Planck equation

$$\partial_t p = D s^c \Delta^{\mu/2} s^{(c-1)} p - D p s^{c-1} \Delta^{\mu/2} s^c, \tag{4}$$

where $p = p(\boldsymbol{x},t), s = s(\boldsymbol{x})$ and D is a generalized diffusion coefficient. The fractional Laplacian is defined by

$$\Delta^{\mu/2} f(\boldsymbol{x}) = \mathcal{C}_\mu \int \mathrm{d}\boldsymbol{y} \frac{f(\boldsymbol{y}) - f(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|^{d+\mu}},$$

with $\mathcal{C}_\mu = 2^\mu \pi^{-n/2} \Gamma((\mu+n)/2)/\Gamma(-\mu/2)$. In Fourier space $\Delta^{\mu/2}$ corresponds to multiplication by $-|\boldsymbol{k}|^\mu$: $\mathcal{F}\{\Delta^{\mu/2} f\}(\boldsymbol{k}) = -|\boldsymbol{k}|^\mu \mathcal{F}\{f\}(\boldsymbol{k})$. Note that the fractional Fokker–Planck equation (4) is equivalent to a number of known stochastic processes for specific choices of the parameters c and μ. For instance when $s(\boldsymbol{x}) = \text{constant}$, (4) is equivalent to free superdiffusion, i.e., (1). When $c = 1/2$ and $\mu = 2$, (4) reads $\partial_t p = -\nabla F\, p + \Delta p$, with $F(\boldsymbol{x}) = -V'(\boldsymbol{x})$ and $V(\boldsymbol{x}) = -\beta^{-1} \log\, s(\boldsymbol{x})$, i.e., diffusion in an external force field. When $c = 0$, (4) reads $\partial_t p = D\Delta^{\mu/2} p/s$, which is equivalent to generalized multiplicative Langevin dynamics for the process $\boldsymbol{X}(t)$, i.e., $\mathrm{d}\boldsymbol{X} = D(\boldsymbol{X})\mathrm{d}L_\mu$ with $D(\boldsymbol{X}) = \sqrt{2}(\exp\, \beta V(\boldsymbol{X})/2)/D$ and $L_\mu(t)$ is a homogeneous Lévy stable process.

In the following, we investigate the relaxation properties of one-dimensional processes governed by the fractional Fokker–Planck equation (4). It is convenient to make a transformation of variables $\psi = s^{1/2} p$ and rewrite the dynamics as a generalized Schrödinger equation $\partial_t \psi = \mathcal{H}\psi$ with a Hamiltonian

$$\mathcal{H}\psi = s^{c-1/2} \Delta^{\mu/2} s^{c-1/2} \psi - \psi s^{c-1} \Delta^{\mu/2} s^c, \tag{5}$$

possessing identical spectral properties.

We model inhomogeneities as salience fields of the type $s = \exp(-\varepsilon v)$, where $v = v(x)$ is a random potential with zero mean and unit variance. The parameter $\varepsilon > 0$ quantifies the strength of the inhomogeneity. For v we chose random phase potentials defined by

$$v(x) = \frac{1}{2\pi}\int \mathrm{d}k\, \phi(k)\, \mathrm{e}^{\mathrm{i}\theta(k)-\mathrm{i}kx}. \tag{6}$$

Here, the phase $\theta(k)$ is uniformly distributed on the interval $(0, 2\pi]$ and the amplitude is given by the correlation spectrum $S(k)$, $\overline{\phi(k)\phi(k')} = 2\pi S(k)\delta(k-k')$, which we chose to be Gaussian with a correlation length ξ: $S(k) = 2\xi \exp(-k^2\xi^2/\pi)$. For small ε we can expand (5) up to the second order and obtain a Hamiltonian of the form $\mathcal{H} = \Delta^{\mu/2} + \hat{U}$ which we can treat perturbation theoretically to obtain a spectrum of the form $E(k) \approx -D_{\mu,c}(k;\varepsilon)|k|^{\mu}$. Here, $D_{\mu,c}(k;\varepsilon)$ describes the relaxation properties on the corresponding scale length $\lambda = k^{-1}$. In the limit of vanishing potential the generalized diffusion coefficient is identical to the diffusion coefficient D of the free system. Typically, a random spatial inhomogeneity slows down a random process, and one generally expects $D_{\mu,c}(k;\varepsilon)$ to be smaller than $D_{\mu,c}(k;\varepsilon = 0)$. Up to second order in ε we obtain

$$D_{\mu,c}(k;\varepsilon)/D = 1 - \varepsilon^2 G_{\mu,c}(k), \tag{7}$$

where $G_{\mu,c}(k) = \frac{1}{2\pi}\int \mathrm{d}q S(q) g_{\mu,c}(k/q)$ and

$$\begin{aligned} g_{\mu,c}(z) = {} & -2(c-1/2)^2 + \frac{1}{z^{\mu}}\{2c(c-1) - (c-1/2)^2(|1-z|^{\mu} + |1+z|^{\mu}) \\ & + [(c-1/2)(z^{\mu} + |1+z|^{\mu}) - c]^2/(|1+z|^{\mu} - z^{\mu}) \\ & + [(c-1/2)(z^{\mu} + |1-z|^{\mu}) - c]^2/(|1-z|^{\mu} - z^{\mu})\}. \end{aligned} \tag{8}$$

The limit $k \to 0$ yields the long scale asymptotics for the process for which we obtain

$$G_{\mu,c}(0) = \begin{cases} 1/2 - c^2, & 0 < \mu < 2, \\ 1/2 + 2c^2, & \mu = 2. \end{cases} \tag{9}$$

This result indicates that with increasing influence of the target salience (increasing values of c) superdiffusive processes ($\mu < 2$) exhibit the opposite behaviour as ordinary diffusion processes ($\mu = 2$). As c is increased $G_{\mu,c}(0)$ increases as well for ordinary diffusion, which means that these processes are attenuated more strongly. Quite contrary to superdiffusive processes, for which a more pronounced target influence decreases $G_{\mu,c}(0)$. This implies that slowing down of superdiffusive processes becomes weaker as the target weight in the rates is increased. Note also that for weak potentials the magnitude of this acceleration is independent of the Lévy exponent μ, see figure 2. Only when $c = 0$, and target salience has no impact on the transition rate, are all processes slowed down by the same amount, i.e., $G_{\mu,0}(0) = 1/2$. The counterintuitive independence of the generalized diffusion coefficient as function of μ is only attained in the infinite system, for disscussion see also [10].

Note that, as c is increased for $\mu < 2$, the function $G_{\mu,c}(0)$ even becomes negative for $c > c_{\mathrm{crit}} = 1/\sqrt{2}$. This implies that these processes are no longer slowed down by the spatial inhomogeneity but rather accelerated, as a negative value for $G_{\mu,c}(0)$ implies a diffusion

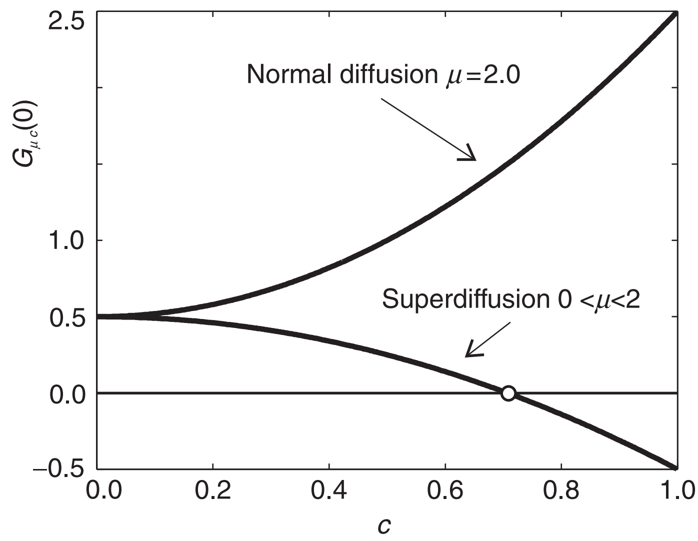

Figure 2. The impact of source and target location on the asymtotics of random walk processes. The generalized diffusion coefficient depends quadratically on the magnitude ε of the inhomogeneity: $D_{\mu,c} = 1 - G_{\mu,c}(0)\varepsilon^2$. The pre-factor $G_{\mu,c}(0)$ quantifies the impact of the spatial inhomogeneity, i.e., when $G_{\mu,c}(0) > 0$ the process is slowed down, when $G_{\mu,c}(0) < 0$ the process is accelerated. The two lines depict $G_{\mu,c}(0)$ as a function of the weight parameter c for ordinary diffusion ($\mu = 2$, upper line) and Lévy flights ($\mu < 2$, lower line). As c is increased (increasing weight of the target location), ordinary diffusion is slowed down more strongly contrary to the Lévy flights for which attenuation decreases until a critical value of $c_{\text{crit}} = 1/\sqrt{2}$ is reached (denoted by small circle). Beyond this point Lévy flights are accelerated by the external inhomogeneity.

coefficient larger than the one for free superdiffusion, i.e., $D_{\mu,c}(k \to 0; \varepsilon) > D$. Consequently, the common notion that random processes are typically slowed down by spatial inhomogeneities is not valid when superdiffusive processes are involved. Unlike ordinary diffusion processes that are trapped in localized regions of high salience, non-local Lévy flights can jump among these regions. With increasing impact of the target location this effective transport can be faster as intermediate regions of low salience are not explored.

For large ε, we computed the generalized diffusion coefficient $D_{\mu,c}(\varepsilon)$ numerically for three periodic and one random phase potential: a cosine potential $v(x) = \sqrt{2}\cos(x/\lambda)$, a potential with localized potential minima and one with localized potential peaks, for which $v(x) = \pm a\cos^{\gamma}(x/\lambda) + b$ ($a, b > 0$, $\gamma = 32$), and a potential defined by (6) with a gaussian spectrum. The results are depicted in figure 3.

For $c = 0$ (i.e. full weight of source location) the processes in a given environment behave similarly, independent of the exponent μ (figure 3(a)), ordinary diffusion and all superdiffusive processes exhibit the same quadratic decrease of $D_{\mu,c}$ with ε in a fixed potential. For $c = 1/2$ (figure 3(b)) only the superdiffusive processes exhibit an identical response to a given potential, the response of the ordinary diffusion process differs. When c is increased beyond the critical value the difference between ordinary diffusion and Lévy flights becomes maximal and changes qualitatively (figure 3(c)). In this regime ordinary diffusion processes are still slowed down by the

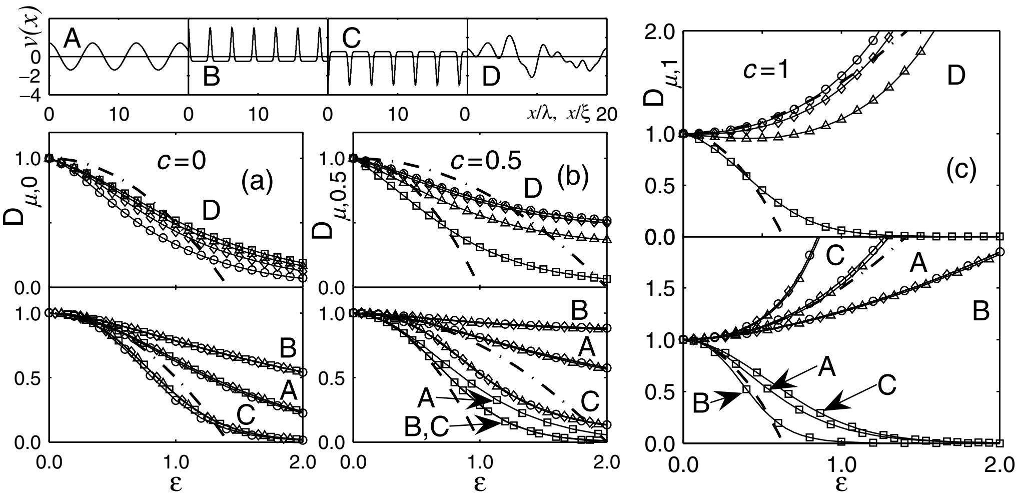

Figure 3. Generalized diffusion coefficient $D_{\mu,c}$ as a function of the magnitude ε of the salience field $s(x) = \exp(-\varepsilon v(x))$. The function $D_{\mu,c}$ is shown for four types of potentials, a cosine potential (A), potentials with localized maxima and minima, (B and C, respectively) and a random phase potential (D). Potentials are displayed on the top left panel. Results for three values of the weight parameter c are presented on the panel (a) $c = 0$ and (b) $c = 0.5$ and (c) $c = 1$. Dashed line and dash-dotted lines are analytical results for $\mu = 2$ and $0 < \mu < 2$ we obtained from perturbation theory (9). Each type of symbol corresponds to one Lévy exponent: ○, $\mu = 0.5$; ◇, $\mu = 1.0$; △, $\mu = 1.5$; □, $\mu = 2.0$. For $c = 1$ in contrast to diffusive processes, which are attenuated, the superdiffusive ones are enhanced by inhomogeneity.

spatial inhomogeneity as opposed to the Lévy flights which exhibits an increase in the diffusion coefficient with increasing potential strength and are thus accelerated. For small values of ε the numerics agree well with our results obtained by the perturbation theory, above (i.e. (9)). Despite the fact that for all values of $\mu < 2$, the curves collapse on one single function, Lévy flights are sensitive to the potential shape. The deviations of the processes in random phase potential with small μ values when $c = 0$ (figure 3(a)) and large μ values when $c = 1$ (figure 3(c)) are due to finite-size effects explained below.

The above considerations were restricted solely to infinite systems. A key question is how these processes behave in finite systems and to what extent finite size effects play a role. Therefore, we investigate the relaxation properties in finite systems of length $2\pi L$, modulated by the periodic potential of the wavelength $2\pi\lambda$. To this end we consider the quantity

$$\delta\tau = \tau/\tau_{\text{free}} - 1, \tag{10}$$

where τ is the relaxation time of the process and τ_{free} is the relaxation time of the same process without spatial inhomogeneity. In a finite system the corresponding Hamiltonian has a discrete spectrum. The continuous Bloch bands (for the infinite system) split into $M = L/\lambda$ discrete eigenvalues $E_n(q_m)$, $n = 0..\infty$, $m = 1..M$, where $q_m = m\lambda/L$ is a Bloch phase. The smallest

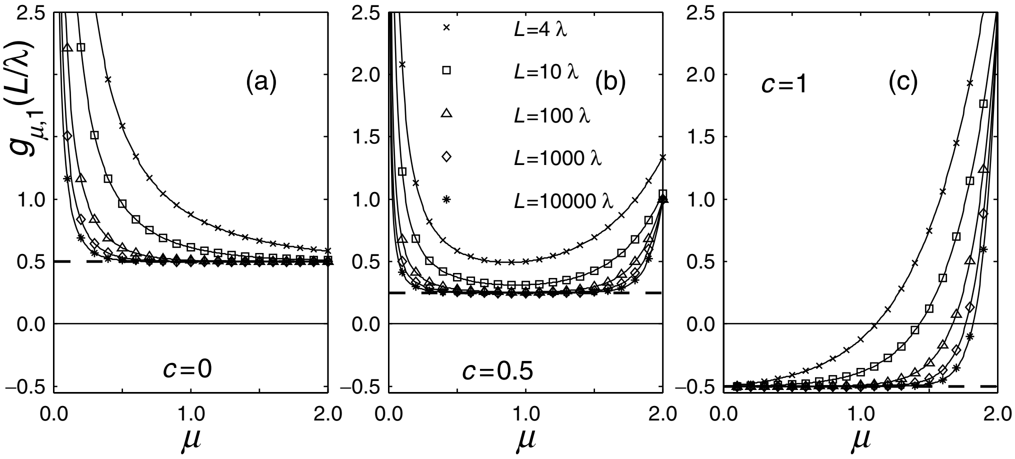

Figure 4. The impact of the heterogeneity on the relaxation time of the processes $\delta t \sim g_{\mu,c}(\lambda/L)$ with different power-law exponents μ, in dependence on the size of the system $2\pi L$ for the cosine potential with period $2\pi\lambda$ and three values of the weight parameter c: (a) $c = 0$ and (b) $c = 0.5$ and (c) $c = 1$. Dashed lines denote the large scale limits, according to (9). With increasing L/λ ratio when $c = 1$ acceleration (negative values of $g_{\mu,c}$) occurs for larger μ exponents.

Bloch phase is given by $q_1 = \lambda/L$. The relaxation time is related to the smallest eigenvalue of the spectrum by $\tau = 1/E_0(\lambda/L)$. Considering (10) and recalling that $\tau_{\text{free}} = L^{\mu}$ we obtain

$$\delta\tau \approx \varepsilon^2 2 \sum_{m>0} g_{\mu,c}(m\lambda/L)|\hat{v}_m|^2,$$

which reads $\delta\tau \sim \varepsilon^2 g_{\mu,c}(\lambda/L)$ for the cosine potential. See figure 4 for three values of $c = 0,\ 0.5$ and 1. Small values of $\delta\tau$ correspond to a small effect of the inhomogeneity. For $c = 0$ (figure 4(a)), the ordinary diffusion process exhibits the smallest value of $\delta\tau$, which implies that in situations in which the source salience is important, diffusion relaxes fastest. On the contrary, for $c = 1$ (figure 4(c)) strongly superdiffusive processes (i.e. $\mu \to 0$) exhibit a small $\delta\tau$. Only when both, source and target possess an equal impact on the jump rates (i.e. $c = 0.5$ (figure 4(b)), $\delta\tau$ exhibits a minimum for intermediate values of the Lévy exponent μ [15]. In finite systems there is a continuous transition from attenuation to acceleration with varying exponent μ. In figure 4(c) one can see, that only Lévy flights with small exponents can be accelerated.

We considered the consequences of the relative weight of source and target locations in one-dimensional random walk processes, evolving in inhomogeneous environment. Our analysis revealed essential differences between superdiffusive Lévy flights and ordinary random walks when they occur in regular and random spatial inhomogeneities. Unlike ordinary random walks, Lévy flights can be accelerated when the influence of the target salience is sufficiently large, which may shed a new light on optimal search strategies in heterogeneous landscapes, and dispersal phenomena in population dynamical systems and various physical and biological contexts.

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